J. Appl. Maths Mechs Vol. 57, No. 2, pp. 237–245, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00
© 1993 Pergamon Press Ltd

DECOMPOSITION OF MOTIONS IN NON-LINEAR SYSTEMS WITH SPINNING PHASE UNDER RANDOM PERTURBATIONS†

A. S. KOVALEVA

Moscow

(*Received* 17 *May* 1991)

A decomposition procedure is proposed for stochastic systems reducible to standard form with a spinning phase. It is proved that the slow motion converges to a diffusion process.

1. WE SHALL study systems whose dynamics are described by the equations

$$x^{\cdot}=\varepsilon F(x,\theta,\xi(t))+\varepsilon^2 G(x,\theta) \tag{1.1}$$

$$\theta^{\cdot}=\omega(x)+\varepsilon H(x,\theta,\xi(t))+\varepsilon^2 D(x,\theta)$$

$$x\in R_n,\quad \theta\in R_1,\quad x(0)=a$$

where $\xi(t)$ is a stochastic process with sample paths in R_l, $F(x, \theta, \xi)$, etc. are deterministic vectors, and ε is a small parameter.

Asymptotic methods have been used to investigate equations of the type (1.1) when $\omega(x)=\omega_0=\text{const}$ and systems are reducible to standard form. It has been proved that under certain conditions (the most general formulation may be found in [1]) the process $x(t, \varepsilon)$ converges weakly as $\varepsilon\to 0$ [2] to a diffusion process $x_0(\tau)$ which is a solution of the stochastic equation

$$dx_0=b(x_0)d\tau+\sigma(x_0)dw,\quad x_0(0)=a \tag{1.2}$$

where $\tau=\varepsilon^2 t$, $w(\tau)$ is an l-dimensional standard Wiener process. A considerable amount of work has been done on a rigorous proving of the passage to the limit from (1.1) to (1.2) for $\omega=\omega_0$; detailed bibliographies may be found in [1, 3, 4]. Similar results have been obtained for equations with spinning phase where $\xi=\xi(\theta)$ [4, 5].

Using an approach developed in [4, 5], we shall establish asymptotic decomposition of motions for (1.1) as $\varepsilon\to 0$.

We shall assume that the coefficients of Eqs (1.1) satisfy the following conditions. The functions F and H may be written in the form

$$F=F_0(x,\theta)\xi(t),\quad H=H_0(x,\theta)\xi(t)$$

where $\xi(t)$ are stationary stochastic processes with zero mean and sample paths in $D^l[0, \infty)$ [2]. The processes $\xi(t)$ are defined in a standard probability space [2, 6]; for brevity, only the dependence on time t will be indicated explicitly; the dependence on the random argument,

†*Prikl. Mat. Mekh.* Vol. 57, No. 2, pp. 22–30, 1993.

characterizing the sample path, will be omitted; F_0, H_0 are matrices of the appropriate dimensions.

The components of the vector process $\xi(t)$ are supposed to satisfy *conditions A*

$$\mathrm{M}\xi(t)=0$$
$$\mathrm{M}|\mathrm{M}_t[\ldots[[\xi(u_1)]^0\xi(u_2)]^0\ldots\xi(u_n)]^0|\leq c_n\alpha_1(u_1-t)\ldots\alpha_n(u_n-u_{n-1})$$
$$0\leq t\leq u_1\ldots\leq u_n,\quad n=1,2,3$$

where $z^0=z-\mathrm{M}z$, $\mathrm{M}z(t)$ is the mathematical expectation and $\mathrm{M}_sz(t)$ is the conditional mathematical expectation of $z(t)$.

The constant c_n depends on $\mathrm{M}|\xi(t)|^r$, and the bounded positive definite functions α_n must satisfy the condition

$$\int_0^\infty u^2\alpha_n(u)du<\infty$$

It has been proved [1, 4] that these conditions hold, in particular, for stationary normal processes and bounded uniformly strong mixing processes with a suitable mixing coefficient [2].

The deterministic coefficients of system (1.1) satisfy *conditions B* in the domain $\theta\in R_1$, $x\in S$, where S is a bounded sphere in R_n, the following conditions hold uniformly in x,θ:

1. F_0, H_0, G and D are periodic or quasi-periodic, and bounded as functions of θ;
2. F_0 is continuous and twice continuously differentiable with respect to x and θ; G is continuous and continuously differentiable with respect to x and θ; G is continuous and continuously differentiable with respect to x; D is bounded;
3. the frequency $\omega(x)\geq\omega_0>0$ is continuous and twice continuously differentiable.

To construct an approximating diffusion process $x_0(\tau)$, we will use the diffusion approximation procedure of [1, 7], adapted to the analysis of fast phase systems as in [4, 5].

We will begin with the necessary definitions [7]. Let $f^\varepsilon(\tau)$ be a scalar stochastic process and $L^\varepsilon f^\varepsilon(\tau)$ be the generating differential operator of the process, defined by

$$L^\varepsilon f^\varepsilon(\tau)=\lim_{\delta\to0_+}\delta^{-1}[\mathrm{M}_\tau f^\varepsilon(\tau+\delta)-f^\varepsilon(\tau)] \tag{1.3}$$

It follows from (1.3) [1, 7] that

$$\mathrm{M}_\tau f^\varepsilon(T)-f^\varepsilon(\tau)=\int_\tau^T\mathrm{M}_\tau L^\varepsilon f^\varepsilon(u)du \tag{1.4}$$

(all equalities are understood in the weak sense [1, 7]).

In particular, if $f^\varepsilon(\tau)=f(x_\varepsilon(\tau))$, where $f(x)$ is a deterministic function and $x_\varepsilon(\tau)$ is a solution of some perturbed system, then (1.4) shows how to evaluate the functional $\mathrm{M}_\tau f(x_\varepsilon(T))$ for paths of the perturbed system.

In particular, if $x_\varepsilon(\tau)=x_0(\tau)$, where $x_0(\tau)$ is a solution of Eq. (1.2), then $L^\varepsilon=L$, where [1, 6, 7]

$$L=b'(x)\frac{\partial}{\partial x}+\frac{1}{2}\mathrm{Tr}A(x)\frac{\partial^2}{\partial x^2},\quad A=\sigma\sigma' \tag{1.5}$$

(throughout this paper, the prime denotes transposition), and (1.4) becomes

$$\mathrm{M}_\tau f(x_0(T))-f(x_0(\tau))=\int_\tau^T\mathrm{M}_\tau Lf(x_0(u))du \tag{1.6}$$

Relations (1.4)–(1.6) indicate a way of calculating and comparing functionals on paths of the perturbed system and the diffusion system.

The asymptotic approach is based on the following assertion [1, 7]. Suppose that the following conditions hold for $\varepsilon \in (0, \varepsilon_0]$, $\tau \in [0, T]$:

1. for any initial $x_0(0) = a \in K$, where K is a compact set in R_n, a unique solution $x_0(\tau) \in D^n[0, \infty)$ of Eq. (1.2) exists;
2. $f(x) \in R_1$ is a sufficiently smooth function with compact support;
3. for every function $f(x)$ and any $T < \infty$, a function $f^\varepsilon(\tau)$ exists in the domain of L^ε such that

$$\sup_{\tau,\varepsilon} \mathrm{M}|f^\varepsilon(\tau)| < \infty \tag{1.7}$$

$$\lim_{\varepsilon\to 0} \mathrm{M}|f^\varepsilon(\tau) - f(x_\varepsilon(\tau))| = 0$$
$$\lim_{\varepsilon\to 0} \mathrm{M}|L^\varepsilon f^\varepsilon(\tau) - Lf(x_\varepsilon(\tau))| = 0 \tag{1.8}$$

where $x_\varepsilon(\tau)$ is a solution of the perturbed system and L is the generating operator (1.5);

4. the sequence $x_\varepsilon(\tau)$ is weakly compact in $D^n[0, \infty)$ [2] and $x_\varepsilon(0) = x_0(0)$.

Then the sequence $x_\varepsilon(\tau)$ converges weakly as $\varepsilon \to 0$ to the diffusion process $x_0(\tau)$ with generating operator L.

It has been shown [1] that conditions (2)–(4) may be weakened, replacing $x_\varepsilon(\tau)$ with a suitable truncated process $x_\varepsilon^N(\tau) = x_\varepsilon(\tau)\eta_N(x_\varepsilon)$ and the functions $f^\varepsilon(\tau)$ and $f(x)$ by truncated functions $f^{\varepsilon N}(\tau)$ and $f^N(x) = f(x)\eta_N(x)$, where $\eta_N(x) = \{1,\ |x| \leqslant N;\ 0,\ x > N\}$. Our assumption that the sequence $x_\varepsilon(\tau)$ is weakly compact justifies the passage to the limit as $N \to \infty$.

2. Relying on these conditions, we shall construct an approximating operator L for system (1.1). The construction is divided into two steps.

Construction of $f^{\varepsilon N}(\tau)$. Let $\tau = \varepsilon^2 t$, $x = x(t, \varepsilon) = x_\varepsilon(\tau)$, $\theta = \theta(t, \varepsilon) = \theta_\varepsilon(\tau)$ be a solution of system (1.1). On the sample path $x(t, \varepsilon)$ we define an arbitrary compactly-supported function $f(x) \in C_3$, which vanishes for $|x| > N$. Define $f^{\varepsilon N}(\tau)$ in terms of $f(x)$ by the formula

$$f^{\varepsilon N}(\tau) = [f(x) + \varepsilon f_1(x,\theta,t) + \varepsilon^2 f_2(x,\theta,t)]\eta_N(x) \tag{2.1}$$

where the coefficients f_1 and f_2 are determined in such a way as to satisfy conditions (1.7) and (1.8).

From (1.1) and (1.3) we obtain

$$L^\varepsilon f^\varepsilon(\tau) = \varepsilon^{-1}(f_x' F + L_t^\varepsilon f_1) + f_x' G + L_t^\varepsilon f_2 \tag{2.2}$$

where L_t^ε is the generating operator

$$L_t^\varepsilon f(x,\theta,t) = \lim_{\Delta\to 0_+} \Delta^{-1}[\mathrm{M}_t f(x(t+\Delta,\varepsilon),\theta(t+\Delta,\varepsilon),t+\Delta) - f(x,\theta,t)] \tag{2.3}$$

Following [1, 4, 5], we construct f_1 so that the coefficient of ε^{-1} in (2.3) vanishes. Define

$$f_1(x,\theta,t) = f_x'(x)\int_t^\infty \mathrm{M}_t F(x,\varphi(u),\xi(u))du \tag{2.4}$$

where

$$\varphi(u) = \varphi^{x,\theta,t}(u) = \theta + \omega(x)(u-t) \tag{2.5}$$

is a solution of the generating system

$$dx/du = 0, \quad d\varphi/du = \omega(x) \tag{2.6}$$

$$x(t) = x, \quad \varphi(t) = \theta, \quad u \geqslant t$$

It has been shown [1, 4] that for functions of type (2.4) the operator L_t^ε can be evaluated by simple differentiation with respect to t, treating the operator M_t as "frozen". The, using (2.5), we get

$$L_t^\varepsilon f_1 = f_{1x} x^{\cdot} + f_{1\theta}\theta^{\cdot} - f_{1\theta}\omega - f_x' F \tag{2.7}$$

where all the terms are evaluated at the point (x, θ, t). This equality relies on the obvious relationships

$$\partial F/\partial\varphi = \partial F/\partial\theta, \quad \partial F/\partial t = -\omega\partial F/\partial\varphi$$

Substituting (2.7) into (2.2) and using (1.1), we obtain

$$L^\varepsilon f^\varepsilon(\tau) = f_{1x}'(F + \varepsilon G) + f_{1\theta}(H + \varepsilon D) + f_x' G + L_t^\varepsilon f_2 \tag{2.8}$$

The function $f_2(x, \theta, t)$ is constructed so as to eliminate the secular components of f_2 in θ. By analogy with previous results [4, 5], we write

$$f_2(x,\theta,t) = \sum_{j=1}^{3}[I_j(x,\theta,t) - S_j(x,\theta)] \tag{2.9}$$

$$I_j = \int_t^\infty [M_t Q_j(x,\varphi(u),u) - MQ_j(x,\varphi(u),u)]du \tag{2.10}$$

where

$$Q_1 = f_{1x}'F, \quad Q_2 = f_{1\theta}H, \quad Q_3 = f_x' G \tag{2.11}$$

It can be shown that for a stationary process $\xi(t)$ the quantities $MQ_j(x, \theta, t)$ are independent of t and

$$MQ_j(x,\theta,t) = q_j(x,\theta)$$

Obviously, $I_3 = 0$ for the deterministic function Q_3. The quantities S_j are given by

$$S_j(x) = \int_0^\theta [q_j(x,\psi) - \bar{q}_j(x)]d\psi \tag{2.12}$$

$$\bar{q}_j(x) = \lim_{\Gamma\to\infty} \frac{1}{\Gamma}\int_0^\Gamma q_j(x,\psi)d\psi \tag{2.13}$$

(it is assumed that the limits exist uniformly in $|x| \leqslant N$).

Thus, the function $f^{\varepsilon N}(\tau)$ is defined by (2.1), (2.4) and (2.9).

Construction of the operator L. It follows from (2.3), (2.8) and (2.9) that

$$L_t^\varepsilon f_2 = -f_{1x}'F - f_{1\theta}H - f_x' G + f_{2x}x^{\cdot} + \sum_{j=1}^{2} I_{j\theta}(\theta^{\cdot} - \omega) + \sum_{j=1}^{3}\bar{q}_j(x) \tag{2.14}$$

$$L^\varepsilon f^{\varepsilon N}(\tau) = \left[\sum_{j=1}^{n}\bar{q}_j(x_\varepsilon(\tau)) + \varepsilon(R_1 + \varepsilon R_2) f(x_\varepsilon(\tau))\right]\eta_N(x_\varepsilon) \tag{2.15}$$

where $R_j f$ are the remainder terms in (2.8) and (2.14), considered as operators acting on f.

We will now express the operators $\bar{q}_j(x)$ in explicit form. It follows from (2.11)–(2.13) that

$$q_1(x,\theta)=\sum_{i=1}^{n}\mathrm{M}f_{1x_i}(x,\theta,t)F^i(x,\theta,\xi(t))$$
$$q_2(x,\theta)=\mathrm{M}f_{1\theta}(x,\theta,t)H(x,\theta,t) \tag{2.16}$$
$$q_3(x,\theta)=\sum_{i=1}^{n}f_{x_i}(x)G^i(x,\theta)$$

where x_i, F^i and G^i are the ith components of the relevant vectors. If f_1 is the function defined by (2.4), then

$$f_{1x_i}=\sum_{j=1}^{n}f_{x_j}(x)\int_t^\infty \mathrm{M}_tF_{x_i}^j(x,\varphi(u),\xi(u))du+\sum_{j=1}^{n}f_{x_ix_j}(x)\int_t^\infty \mathrm{M}_tF^j(x,\varphi(u),\xi(u))du \tag{2.17}$$

Using (2.5), we write

$$F_{x_i}^j(x,\varphi(u),\xi(u))=[F_{z_i}^j(z,\theta+\omega(x)(u-t),\xi(u))]_{z=x}+$$
$$+F_\theta^j(x,\theta+\omega(x)(u-t),\xi(u))\omega_{x_i}(x)(u-t) \tag{2.18}$$

In exactly the same way

$$f_{1\theta}(x,\theta,t)=\int_t^\infty \mathrm{M}_tF_\theta(x,\theta+\omega(x)(u-t),\ \ \xi(u))du \tag{2.19}$$

We now substitute (2.17)–(2.19) into (2.16) and average as in (2.13). Using the well-known property of the mathematical expectation $\mathrm{MM}_t=\mathrm{M}$ [6], we obtain the final result

$$\bar{q}_1(x)=[b_1'(x)+b_3'(x)]f_x(x)+\tfrac{1}{2}\mathrm{Tr}A(x)f_{xx}(x) \tag{2.20}$$
$$\bar{q}_2(x)=b_2'(x)f_x(x),\quad \bar{q}_3(x)=\bar{G}'(x)f_x(x)$$

$$\bar{G}(x)=\lim_{\Gamma\to\infty}\frac{1}{\Gamma}\int_0^\Gamma G(x,\theta)d\theta \tag{2.21}$$

$$b_j(x)=\lim_{\Gamma\to\infty}\frac{1}{\Gamma}\int_0^\Gamma d\theta\int_0^\infty B_j(x,\theta,s)ds,\quad j=1,2,3$$

$$B_1^i(x,\theta,s)=\sum_{j=1}^{n}\mathrm{M}[F_{z_j}^i(z,\theta+\omega(x)s,\xi(t+s))F_j(x,\theta,\xi(s))]_{z=x} \tag{2.22}$$

$$B_2^i(x,\theta,s)=\mathrm{M}[F_\theta^i(x,\theta+\omega(x)s,\xi(t+s))H(x,\theta,s)]$$

$$B_3^i(x,\theta,s)=s\sum_{j=1}^{n}\mathrm{M}[F_\theta^i(x,\theta+\omega(x)s,\xi(t+s))\omega_{x_j}(x)F_j(x,\theta,\xi(t))]$$

$$A(x)=a(x)+a'(x)=\sigma(x)\sigma'(x) \tag{2.23}$$

$$a(x)=\lim_{\Gamma\to\infty}\frac{1}{\Gamma}\int_0^\Gamma d\theta\int_0^\infty \alpha(x,\theta,s)ds$$

$$\alpha_{ij}(x,\theta,s)=\mathrm{M}[F^i(x,\theta+\omega(x)s,\xi(t+s))F^j(x,\theta,\xi(t))]$$

The fact that the integrands of (2.22) and (2.23) are independent of t follows from the stationarity of $\xi(t)$.

Define the operator L as

$$L = b'(x)\frac{\partial}{\partial x} + \frac{1}{2}\mathrm{Tr}A(x)\frac{\partial^2}{\partial x^2}, \quad b = \sum_{j=1}^{3} b_j \tag{2.24}$$

Obviously if $\omega = \omega_0 = \mathrm{const}$, then $b_3 = 0$ and the expression (2.21) is identical—apart from the notation—with the standard formulae [3, 4].

To prove that conditions (1.7) and (1.8) hold, it will suffice to show that if $|x| \leq N$, $t \in R_1$, $\theta \in R_1$, then for all $f(x) \in C_3$

$$\mathrm{M}|f_j(x,\theta,t)| < \infty, \quad \mathrm{M}|R_j f(x)| < \infty, \quad j = 1,2 \tag{2.25}$$

As similar estimates have been established in [4, 5], we will omit the proof. We note that inequalities (2.25) follow from conditions A and B (see above).

The validity of conditions (1.7) and (1.8) also implies that the sequence $x_\varepsilon^N(\tau)$ is weakly compact [1].

If conditions A and B are satisfied and the limits (2.22) and (2.23) exist, then the coefficients of the operator (2.24) satisfy the conditions $b(x) \in C_1$, $a(x) \in C_2$. Consequently [6], a diffusion process $x_0(\tau)$ exists with generating operator (2.24).

We have thus proved the following theorem.

Theorem 1. Suppose that the coefficients of system (1.1) satisfy conditions A and B in the domain $x \in S$, $\theta \in R_1$, $t \in R_1$ and that the limits (2.22) and (2.23) exist there.

Then for $\tau \in [0, T]$ the process $x(t, \varepsilon) = x_\varepsilon(\tau)$ converges weakly as $\varepsilon \to 0$ to a diffusion process $x_0(\tau)$ with generating operator (2.24).

Remark. It is obvious that these results also remain valid in the case when $\omega = \omega(\tau)$. The "slow" time τ may be treated as an additional variable

$$x_{n+1} = \tau, \quad x_{n+1}^{\cdot} = \varepsilon^2, \quad x_{n+1}(0) = 0$$

When that is done the coefficients of the operator (2.24) are defined by (2.20)–(2.23), but $b_3 = 0$.

3. We will now consider a few examples.

Oscillations of a pendulum of variable length suspended from a vertically vibrating point (*Fig*. 1). When the dissipation and perturbation are small and the length varies slowly, the equations of the pendulum reduce to

$$\frac{d}{dt}\left[l^2(\tau)\frac{dz}{dt}\right] + 2\varepsilon^2 n\frac{d}{dt}[l(\tau)z] + l(\tau)[g + \varepsilon\xi(t)]z = 0 \tag{3.1}$$

$$z(0) = \zeta, \quad z^{\cdot}(0) = \nu$$

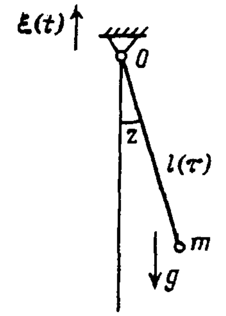

FIG. 1.

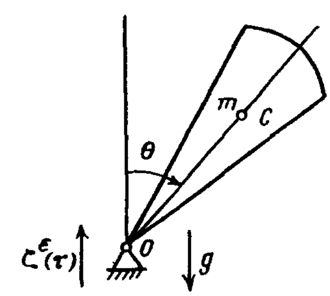

FIG. 2.

where z is the angular deviation of the pendulum from the vertical, $l(\tau)$ is the slowly varying length, $\tau=\varepsilon^2 t$, $\xi(t)$ is the acceleration of the point of suspension—a stationary normal process with zero mean and spectral density $S(\omega)$ and ε is a small parameter.

The change of variables

$$z=x\cos\theta,\quad z^{\cdot}=-\omega(\tau)x\sin\theta \qquad (3.2)$$

$$\omega(\tau)=[g/l(\tau)]^{1/2}$$

reduces Eq. (3.1) to standard form [8]

$$x^{\cdot}=\varepsilon\frac{\xi(t)}{\omega l}x\cos\theta\sin\theta-\varepsilon^2\left[\frac{2}{l}(l_\tau+n)+\frac{\omega_\tau}{\omega}\right]x\sin^2\theta \qquad (3.3)$$

$$\theta^{\cdot}=\omega+\varepsilon\frac{\xi(t)}{\omega l}\cos^2\theta-\varepsilon^2\left[\frac{2}{l}(l_\tau+n)+\frac{\omega_\tau}{\omega}\right]\sin\theta\cos\theta$$

$$l_\tau=dl/d\tau,\quad \omega_\tau=d\omega/d\tau,\quad x(0)=\alpha,\quad \theta(0)=\beta$$

In this case

$$F=(2\omega l)^{-1}\xi(t)x\sin 2\theta$$
$$H=(2\omega l)^{-1}\xi(t)(1+\cos 2\theta) \qquad (3.4)$$
$$G=-(2l)^{-1}(3l_\tau+4n)x\sin^2\theta$$

Substituting (3.4) into (2.21)–(2.23), we obtain

$$b_1=\frac{x}{16}D^2,\quad b_2=\frac{x}{8}D^2$$
$$\overline{G}=-\left(\frac{3}{4}\frac{l_\tau}{l}+\frac{n}{l}\right)x \qquad (3.5)$$
$$A_{11}=\sigma^2=\frac{x^2}{8}D^2,\quad D^2=\frac{S(2\omega)}{(\omega l)^2}$$

It follows from Theorem 1 that the process $x(t,\varepsilon)=x_\varepsilon(\tau)$ converges weakly as $\varepsilon\to 0$ to a diffusion process $x_0(\tau)$ satisfying the equation

$$dx_0=b_0x_0d\tau+\sigma_0x_0dw,\quad x_0(0)=\alpha$$
$$b_0=\frac{3}{16}D^2-\left(\frac{3}{4}\frac{l_\tau}{l}+\frac{n}{l}\right),\quad \sigma_0^2=\frac{1}{8}D^2 \qquad (3.6)$$

We shall estimate the root-mean-square amplitude of the oscillations, $m_\varepsilon=\mathrm{M}x_\varepsilon^2$. Since the process is weakly convergent, it follows that over a time interval $0\leqslant\tau\leqslant T$ the process $m_\varepsilon(\tau)$ remains in the ε-neighbourhood of the function $m_0(\tau)=\mathrm{M}x_0^2(\tau)$ defined as the solution of the equation [6]

$$dm_0/d\tau=[2b_0(\tau)+\sigma_0^2(\tau)]m_0,\quad m_0(0)=\alpha^2 \qquad (3.7)$$

$$m_0(\tau)=\alpha^2\exp\left\{\int_0^\tau[2b_0(s)+\sigma_0^2(s)]ds\right\} \qquad (3.8)$$

It follows from (3.6) and (3.8) that the solution of system (3.7) is asymptotically stable if

$$\int_0^\tau \left[\frac{D^2(s)}{2} - \frac{2n}{l(s)}\right] ds < \frac{3}{2} \ln \frac{l(\tau)}{l(0)} \tag{3.9}$$

If this condition is satisfied, $m_\varepsilon(\tau)$ remains in the ε-neighbourhood of the exponentially decreasing function $m_0(\tau)$.

Condition (3.9) may be replaced by the stronger, but more easily verified, condition $2b_0(\tau)+\sigma_0^2(\tau)<0$ for all $\tau>0$, i.e.

$$S[2\omega(\tau)] < [\omega(\tau)l(\tau)]^2[4n+3l_\tau(\tau)] \tag{3.10}$$

If $l_\tau = 0$, $\omega = \text{const}$, condition (3.10) reduces to a well-known condition for root-mean-square stability [3].

Fast rotation of a plane pendulum with randomly vibrating axis of suspension (*Fig*. 2). The equation of rotation of the pendulum in the vertical plane is

$$\begin{gathered} Jd^2\theta / d\tau^2 - ml(g + d^2\zeta^\varepsilon / d\tau^2)\sin\theta = 0 \\ \tau = 0, \quad \theta = 0, \quad d\theta / d\tau = \gamma^\varepsilon \end{gathered} \tag{3.11}$$

where J is the moment of inertia about the axis of rotation O, m is the mass of the pendulum, l is the arm OC, and $\zeta^\varepsilon(\tau)$ is the vertical displacement of the point of suspension. It is assumed that $\zeta^\varepsilon(\tau)$ is a small but fast perturbation, so that

$$mlJ^{-1}\zeta^\varepsilon(\tau) = \varepsilon\zeta(\tau/\varepsilon^2), \quad \varepsilon \ll 1$$

Setting $\tau/\varepsilon^2 = t$, we write

$$\begin{gathered} \theta^{\cdot\cdot} = \varepsilon^2(\lambda^2 + \varepsilon^{-1}\zeta^{\cdot\cdot}(t))\sin\theta \\ t = 0, \quad \theta = 0, \quad \theta^{\cdot} = \gamma \end{gathered} \tag{3.12}$$

where $\lambda^2 = mlgJ^{-1}$, $\gamma = \varepsilon\gamma^\varepsilon$.

We are considering a situation with fast rotation: $\gamma^\varepsilon \gg \lambda$.

Reducing (3.12) to standard form, we obtain

$$\begin{gathered} x^{\cdot} = \varepsilon\xi(t)\sin\theta + \varepsilon^2 G(\theta), \quad x(0) = \gamma \\ \theta^{\cdot} = x, \quad \theta(0) = 0 \end{gathered} \tag{3.13}$$

System (3.13) has the form of (1.1) with

$$F = \xi(t)\sin\theta, \quad G = \lambda^2 \sin\theta, \quad H = 0, \quad \omega = x, \quad \xi(t) = \zeta^{\cdot\cdot}(t) \tag{3.14}$$

Consequently, $b_1 = b_2 = 0$, $\overline{G} = 0$, $b_3 = \frac{1}{4}S_x(x)$, $\sigma^2 = \frac{1}{2}S(x)$, where $S(\omega)$ is the spectral density of $\xi(t)$. Thus, $x(t, \varepsilon)$ converges weakly as $\varepsilon \to 0$ to the diffusion process $x_0(\tau)$ with the generating operator

$$L = b_3(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} = \frac{1}{4}\frac{\partial}{\partial x}\left(S(x)\frac{\partial}{\partial x}\right) \tag{3.15}$$

We will estimate the residence time τ_ε when the process $x_\varepsilon(\tau)$ stays in the α-neighbourhood of the stationary solution $x = \gamma$. It follows from the weak convergence condition that $M\tau_\varepsilon \to M\tau_0$ as $\varepsilon \to 0$, where τ_0 is the similar residence time for the process $x_0(\tau)$. In turn [6], $M\tau_0 = V(\gamma)$, where $V(x)$ is a solution of the equation

$$LV(x)=-1,\quad V[\gamma(1+\alpha)]=V[\gamma(1-\alpha)]=0 \tag{3.16}$$

L being the operator (3.15). Equation (3.16) is solvable by quadratures. To emphasize the physical meaning of the solution, let us assume that α is fairly small. Then

$$\mathrm{M}\tau_0=V(\gamma)=2\alpha^2\gamma^2/S(\gamma) \tag{3.17}$$

The quantity τ_0 could be considered as a measure of the closeness of the motion to steady-state motion: the longer the system stays in the neighbourhood of a stationary point, the closer the solution is to steady-state. In particular, $\mathrm{M}\tau_0\to\infty$ as $\xi\to 0$. In turn, it follows from (3.17) that $\mathrm{M}\tau_0$ decreases as $S(\gamma)$ increases and $\mathrm{M}\tau_0\to 0$ as $S(\gamma)\to\infty$, i.e. the system shows a "resonance" acceleration effect.

REFERENCES

1. KUSHNER H. J., *Approximation and Weak Convergence Methods for Random Processes with Applications to Stochastic Systems Theory.* MIT Press, Cambridge, MA, 1984.
2. BILLINGSLEY P., *Convergence of Probability Measures.* John Wiley, New York, 1968.
3. DIMENTBERG M. F., *Non-linear Stochastic Problems of Mechanical Oscillations.* Nauka, Moscow, 1980.
4. KOVALEVA A. S., *Control of Oscillatory and Vibro-impact Systems.* Nauka, Moscow, 1990.
5. KOVALEVA A. S., The separation of motions in non-linear oscillatory systems with random perturbation. *Prikl. Mat. Mekh.* **54**, 530–536, 1990.
6. GIKHMAN I. I. and SKOROKHOD A. V., *Introduction to the Theory of Random Processes.* Nauka, Moscow, 1977.
7. ETHIER S. and KURTZ T., *Markov Processes: Characterization and Convergence.* John Wiley, New York, 1986.
8. MITROPOL'SKII Yu. A., *The Averaging Method in Non-linear Mechanics.* Naukova Dumka, Kiev, 1971.

Translated by D.L.